Math [redacted]

**Sample Final Exam**

Name: KEY

**Instructions:**

- All answers must be written clearly.
- You may use a calculator, but you must show all your work in order to receive credit.
- Be sure to erase or cross out any work that you do not want graded.
- If you need extra space, you may use the back sides of the exam pages (if you do, please write me a note so that I know where to look).
- You must include all work to receive full credit.

1. Consider a standard deck of 52 cards. What is the probability of a four of a kind? (This occurs when the cards have denominations $a, a, a, a, b$.)

$$\binom{13}{1}\cdot\binom{4}{4}\cdot\binom{12}{1}\binom{4}{1}\Big/\binom{52}{5}$$

4 of Kind Rank | Choose all 4 | Choose rank | Choose suit

$$= \frac{13\cdot 12\cdot 4}{\binom{52}{5}}$$

2. Consider a roullete wheel consisting of 50 numbers 1 through 50, 0, and 00. If Phan always bets that the outcome will be one of the numbers 1 through 20, what is the probability that

(a) Phan will lose his first 7 bets,

# are 1 - 50, 0, 00 = 52 total

$$p = \frac{20}{52}$$

$$\mathbb{P}(\text{lose } 1^{st}\ 7 \text{ bets}) = \left(\frac{32}{52}\right)^7 \approx .033$$

(b) his first win will occur on his ninth bet?

$$X \sim \text{Geometric}\left(\frac{20}{52}\right)$$

$$\mathbb{P}(X=9) = \left(\frac{32}{52}\right)^{9-1}\left(\frac{20}{52}\right) = \left(\frac{32}{52}\right)^{8}\left(\frac{20}{52}\right)$$

$$\approx .0079$$

3. The monthly worldwide average number of airplane crashes of commercial airlines is 3.5. What is the probability that at most 1 accident will occur in next 2 months?

2 month average is $= 2(3.5) = 7$

$X \sim \text{Poisson}(7)$, $X =$ # of accidents in a 2 month period.

$\mathbb{P}(X = n) = e^{-7} \frac{7^n}{n!}$

$\mathbb{P}(X \leq 1) = \mathbb{P}(X = 0) + \mathbb{P}(X = 1)$

$= e^{-7} + e^{-7} \frac{7^1}{1!} \approx \boxed{.00723}$

4. The r.v. $X$ has a mgf given by

$$m_X(t) = \frac{1}{1-t}, \quad t < 1.$$

If $u$ is some unknown number greater than 0, what is $\mathbb{P}(X > 1 + u \mid X > u)$?

- Note that by matching the mgf of $X$ with the known ones in the table we see that $X \sim \exp(1)$, since $\frac{1}{1-t} = \frac{\lambda}{\lambda - t}$.
- From class, we know $\mathbb{P}(X > a) = e^{-\lambda a} = e^{-a}$
- Thus,

$\mathbb{P}(X > 1 + u \mid X > u) = \frac{\mathbb{P}(X > 1 + u, X > u)}{\mathbb{P}(X > u)} = \frac{\mathbb{P}(X > 1 + u)}{\mathbb{P}(X > u)}$

$= \frac{e^{-(1+u)}}{e^{-u}} = \boxed{e^{-1}}$, Recall that this means $X$ is memoryless.

5. A manufacturing company sources widgets from three different suppliers (A, B, and C). Based on the company's quality control data, it appears that 3 percent of widgets coming from A are faulty, as are 5 percent of the widgets coming from B, and 2 percent coming from C. Based on recent purchasing records, suppliers A, B, and C supply 30 percent, 20 percent, and 50 percent of the company's stock of widgets, respectively.

(a) What is the probability that a random widget from the company's stock is faulty?

$\mathbb{P}(A) = .30$, $\mathbb{P}(B) = .20$, $\mathbb{P}(C) = .5$

$F$ = Faulty, By the Law of total probability,

$$\mathbb{P}(F) = \mathbb{P}(F|A)\mathbb{P}(A) + \mathbb{P}(F|B)\mathbb{P}(B) + \mathbb{P}(F|C)\mathbb{P}(C)$$
$$= (.03)(.3) + (.05)(.2) + (.02)(.5)$$
$$= \boxed{.029}$$

(b) Given that a widget is faulty, what is the probability that it came from supplier C?

$$\mathbb{P}(C|F) = \frac{\mathbb{P}(C \cap F)}{\mathbb{P}(F)} = \frac{\mathbb{P}(F|C)\mathbb{P}(C)}{\mathbb{P}(F)}$$

$$= \frac{(.02)(.5)}{.029} = \boxed{.345}$$

(c) Using the definition of independence of events, determine whether the events ~~A~~ F = {widget is faulty} and ~~B~~ C = {widget came from supplier C} are independent or not.

Not independent, because independent means $\mathbb{P}(C|F) = \mathbb{P}(C)$

but $.345 \neq .5$.

6. Suppose the joint density function of the random variables $X$ and $Y$ is

$$f(x,y) = \begin{cases} c(x+y) & 0 < x, y < 1 \\ 0 & \text{otherwise} \end{cases}.$$

(a) Find the value of $c$.

$$1 = \iint_D f(x,y)\,dy\,dx = \int_0^1\int_0^1 c(x+y)\,dy\,dx$$

$$= c\int_0^1 xy + \frac{y^2}{2}\Big|_{y=0}^{y=1} dx = c\int_0^1 \left(x+\frac{1}{2}\right) dx$$

$$= c\left[\frac{x^2}{2} + \frac{1}{2}x\right]_0^1 = c\left[\frac{1}{2}+\frac{1}{2}\right] = c, \Rightarrow \boxed{c=1}$$

(b) Compute $\mathbb{P}(X^2+Y^2 \le 1)$

Domain

$$\mathbb{P}(X^2+Y^2\le 1) = \int_0^1\int_0^{\sqrt{1-x^2}} x+y\,dy\,dx$$

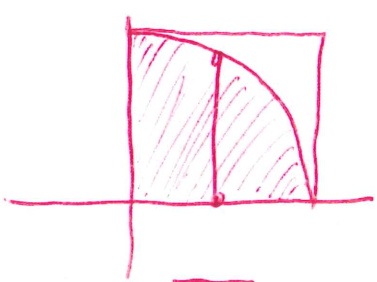
$$= \int_0^1 \left[xy + \frac{y^2}{2}\right]_{y=0}^{y=\sqrt{1-x^2}} dx$$

$X^2+Y^2 \le 1 \to$ Circle

$$= \int_0^1 x\sqrt{1-x^2} + \frac{1-x^2}{2}\,dx$$

u-sub: $u = 1-x^2$, $du = -2x\,dx$, $-\frac{du}{2} = x\,dx$

$$= -\frac{1}{2}\frac{2}{3}(1-x^2)^{3/2} + \frac{1}{2}\left(x - \frac{x^3}{3}\right)\Big|_0^1$$



$$= \left[0 + \frac{1}{2}\left(1-\frac{1}{3}\right)\right] - \left[-\frac{1}{2}\cdot\frac{2}{3} + 0\right]$$

Top $y = \sqrt{1-x^2}$

Bottom $y = 0$

$0 \le x \le 1$

$$= \frac{1}{2}\cdot\frac{2}{3} + \frac{1}{3} = \boxed{\frac{2}{3}}$$

(c) Compute $\mathbb{E}\left[X^2Y\right]$.

$$\mathbb{E}[X^2Y] = \int_0^1\int_0^1 x^2y\,(x+y)\,dy\,dx = \int_0^1\int_0^1 (x^3y + x^2y^2)\,dy\,dx$$

$$= \int_0^1 \left[x^3\frac{y^2}{2} + x^2\frac{y^3}{3}\right]_{y=0}^{y=1} dx$$

$$= \int_0^1 \left[\frac{x^3}{2} + \frac{x^2}{3}\right] dx$$

$$= \frac{x^4}{8} + \frac{x^3}{9}\Big|_0^1 = \frac{1}{8} + \frac{1}{9} = \boxed{\frac{17}{72}}$$

7. Suppose $X$ is a normal r.v. with mean 1 and variance 1 and let Y be an independent Poisson r.v. with parameter 2. What is $\text{Var}\,(2X - Y)$?

Recall $\quad \text{Var}(V+W) = \text{Var}(V) + \text{Var}(W) + 2\,\text{Cov}(V,W)$

$\quad\quad\quad \text{Var}(aW) = a^2\,\text{Var}(W)$

Since $\quad X \sim N(1,1)$, then $\text{Var}\,X = 1$

$\quad\quad\quad Y \sim \text{Poisson}(2)$, then $\text{Var}(Y) = 2$

Thus $\quad \text{Var}(2X - Y) = \text{Var}(2X) + \text{Var}(-Y) + 2\,\text{Cov}(2X, -Y)$

$$= 2^2\,\text{Var}(X) + (-1)^2\,\text{Var}(Y) + 2(2)(-1)\,\text{Cov}(X,Y)$$

$$= 4\cdot 1 + 1\cdot 2 + 0 = \boxed{6}$$

Where I used the fact that $\text{Cov}(X,Y) = 0$, since $X, Y$ independent.

8. Let $X$ be a uniform random variable over $(1, 6)$. Find the moment generating function of $X$.

$$f_X(x) = \begin{cases} \frac{1}{6-1} & 1 < x < 6 \\ 0 & o/w \end{cases} = \begin{cases} \frac{1}{5} & 1 < x < 6 \\ 0 & o/w \end{cases}$$

$$\text{So } m_X(t) = \mathbb{E}[e^{tX}] = \int_1^6 e^{tx} f_X(x)\,dx$$

$$= \int_1^6 e^{tx} \frac{1}{5}\,dx = \frac{1}{5}\left[\frac{1}{t} e^{tx}\right]_{x=1}^{x=6}$$

$$= \boxed{\frac{e^{6t} - e^{t}}{5t}}$$

9. Suppose $X$ has the following moment generating function

$$m_X(t) = \frac{e^t}{1 - t^2}.$$

Find $\mathbb{E}[X]$. (This distribution is known as the *Laplace* distribution)

$$\mathbb{E}X = m'(0) = \frac{d}{dt}\left[\frac{e^t}{1-t^2}\right]_{t=0}$$

$$= \frac{(1-t^2)e^t - e^t(-2t)}{(1-t^2)^2}\Bigg|_{t=0}$$

$$= \frac{1 - e^0 \cdot 0}{1} = \boxed{1}$$

10 9. A person has 100 light bulbs whose lifetimes are independent exponentials with mean 5 hours. If the bulbs are used one at a time, with a failed bulb being replaced immediately by a new one, approximate the probability that there is still a working light bulb after 525 hours.

• Let $X_1, X_2, \ldots, X_{100}$ be the lives (in hours) of the 100 light bulbs.

• Note that $X_i \sim \exp\left(\frac{1}{5}\right)$, since $\boxed{\mu = \frac{1}{\lambda}} \rightsquigarrow \boxed{\lambda = \frac{1}{5}}$

$\hookrightarrow \boxed{\mu = 5}$, $\sigma^2 = Var(X) = 5^2 = 25$ $\boxed{\sigma = 5}$

• $n\mu = 100 \cdot 5 = 500$

• $\sqrt{n}\sigma = \sqrt{100} \cdot 5 = 10 \cdot 5 = 50$

• $\mathbb{P}\left(\begin{matrix}\text{exists working light bulb} \\ \text{after 525 hours}\end{matrix}\right) = \mathbb{P}(X_1 + \cdots + X_{100} > 525)$

$= \mathbb{P}\left(\begin{matrix}\text{lifetime of all} \\ \text{bulbs}\end{matrix} > 525\right)$

$= \mathbb{P}\left(\frac{X_1 + \cdots + X_{100} - 500}{50} > \frac{525 - 500}{50}\right)$

by CLT we have,

$\approx \mathbb{P}\left(Z > \frac{25}{50}\right) = \mathbb{P}(Z > .5) = 1 - \mathbb{P}(Z \leq .5)$

$= 1 - .6915 = \boxed{.3085}$